

Algebraic solution of the Lindblad equation for a collection of multilevel systems coupled to independent environments

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Abstract. We consider the Lindblad equation for a collection of multilevel systems coupled to independent environments. The equation is symmetric under the exchange of the labels associated with each system and thus the open-system dynamics takes place in the permutation-symmetric subspace of the operator space. The dimension of this space grows polynomially with the number of systems. We construct a basis of this space and a set of superoperators whose action on this basis is easily specified. For a given number of levels, M , these superoperators are written in terms of a bosonic realization of the generators of the Lie algebra $\mathfrak{sl}(M^2)$. In some cases, these results enable finding an analytic solution of the master equation using known Lie-algebraic methods. To demonstrate this, we obtain an analytic expression for the state operator of a collection of three-level atoms coupled to independent radiation baths. When analytic solutions are difficult to find, the basis and the superoperators can be used to considerably reduce the computational resources required for simulations.

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1. Introduction

The theory of open quantum systems was developed in order to account for dissipation and loss of coherence in quantum systems. This framework considers a quantum system as composed of two interacting parts: a small, experimentally accessible *open quantum system* and a large *environment*. Many realistic situations are accurately described considering that the environment has short correlation times [1]. The resulting Markovian dynamics is modelled with the *Lindblad master equation* [2, 3], which describes the evolution of the reduced state operator, $\hat{\rho}$, of the open system:

$$\dot{\hat{\rho}} = \frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \frac{1}{2} \sum_{\alpha} \gamma_{\alpha} (2\hat{L}_{\alpha}\hat{\rho}\hat{L}_{\alpha}^{\dagger} - \hat{L}_{\alpha}^{\dagger}\hat{L}_{\alpha}\hat{\rho} - \hat{\rho}\hat{L}_{\alpha}^{\dagger}\hat{L}_{\alpha}). \quad (1)$$

In (1) \hat{H} is the effective Hamiltonian of the system, which differs from the free evolution due to the coupling to the environment, and the Lindblad “jump” operators \hat{L}_{α} are

related to decoherence and dissipation processes that occur with rate γ_α . The dagger (\dagger) denotes Hermitian conjugation. Since the *Liouvillian* \tilde{L} acts on an operator, it is called a *superoperator*. Throughout this article, we will denote superoperators with a breve (\breve{A}).

In many cases finding an analytic solution of (1) is not feasible and one must resort to a numerical solution. In general, the simulation of the Lindblad equation is computationally expensive. However, there are some ways to reduce this cost. The quantum Monte Carlo wavefunction method [4–6] maps the problem of simulating (1) to a Monte Carlo calculation of quantum trajectories in state space, which is amenable to parallel computation. Recently a method was introduced [7] that consists in approximating the evolution of an $n \times n$ density matrix restricting the dynamics to a set of density matrices of rank $m < n$. For many-body systems with short-range interactions, it has been shown that the computational cost of simulating the evolution of observables with a finite spatial support is independent of the system size [8]. The fact that a symmetric Lindblad equation preserves the symmetry of the state operator can be exploited to considerably reduce the computational cost of numerically solving the equation. It has long been known [9] that the symmetric state operator of a collection of two-level systems has only $(N+1)(N+2)(N+3)/6$ independent coefficients. This result was rediscovered in [10] and has been exploited in recent research [11–14]. Here, it will be shown that the symmetric state operator of N M -level systems is characterized by $\text{poly}(N)$ parameters, thus allowing an efficient simulation of the dynamics.

In recent years, several methods have been developed in order to obtain an analytic solution of (1). These include diagonalization of the Liouvillian [15–17], Lie-algebraic methods [18–21], extending the open system with an auxiliary system [22], decomposing the reduced state operator in a sum of diagonal block operators [23] and series expansions [24]. However, exact solutions for a collection of systems are scarce. Recently [10], Hartmann obtained an analytic expression for the state operator of a collection of two-level atoms coupled to independent radiation baths. This involved constructing a basis for the symmetric subspace of the operator space and writing the master equation in terms of a set of superoperators whose action on the above basis vectors can be easily calculated.

The present work extends the results of [10] for a collection of N M -level systems. First, we build a basis for the symmetric subspace of the operator space and show that its dimension grows polynomially with N . Then, we write the Lindblad equation in terms of a bosonic realization of the generators of $\mathfrak{sl}(M^2)$. The action of such superoperators on the basis vectors found above is easily calculated. These results can be used to find an analytic solution of the Lindblad equation using known Lie-algebraic methods. As an example, we obtain an exact analytic expression for the state operator of a collection of three-level atoms coupled to independent radiation baths.

The outline of the article is as follows. In section 2 we present the basics of superoperators and their matrix representation. In section 3 we briefly review the Lie-algebraic method that will be used for solving the Lindblad master equation. In section 4 we build a basis for the symmetric subspace of the operator space and show that its dimension grows polynomially with N . In section 5 we consider the Liouvillian for a collection of N M -level systems subject to independent dissipative processes and we show that it can be written in terms of a bosonic realization of the generators of $\mathfrak{sl}(M^2)$. In section 6 we find an analytic expression for the state operator of a

collection of three-level atoms coupled to independent radiation baths and point out some further applications of the results of this work.

2. Liouville space and superoperators

Let the Hilbert space associated with a single M -level system be \mathcal{H}_M . Orthonormal basis vectors are given, in Dirac's notation, by $|1\rangle, |2\rangle, \dots, |M\rangle$. The state operator $\hat{\rho}$ of the system, as well as any operator \hat{A} acting on \mathcal{H}_M , are elements of a larger Hilbert space $\mathcal{L}_{M^2} = \mathcal{H}_M \otimes \mathcal{H}_M^*$ of dimension M^2 called *Liouville space* or *von Neumann space* [25–27]. The asterisk denotes the dual space and \otimes is the tensor product of vector spaces. For convenience, we introduce a Dirac-like notation to denote ket vectors as $|\hat{A}\rangle \in \mathcal{L}_{M^2}$ and bra vectors as $\langle \hat{A}| \in \mathcal{L}_{M^2}^*$. \mathcal{L}_{M^2} is equipped with the Hilbert–Schmidt scalar product $(\hat{A}|\hat{B}) = \text{Tr}(\hat{A}^\dagger \hat{B})$, which determines the Hilbert–Schmidt norm $\|\hat{A}\|_2 = \sqrt{(\hat{A}|\hat{A})}$. The basis vectors of \mathcal{H}_M induce an orthonormal basis for \mathcal{L}_{M^2} given by the ket-bra operators $|m\rangle\langle n|$ with $m, n \in (1, 2, \dots, M)$, which we denote as $|mn\rangle := ||m\rangle\langle n|$. In terms of this basis we define two matrix representations of an element of \mathcal{L}_{M^2} given by:

$$\begin{aligned} |\hat{A}\rangle &= \sum_{m,n=1}^M (mn|\hat{A})|mn\rangle, & A_{mn} &:= (mn|\hat{A}), \\ |\hat{A}\rangle &= \sum_{\alpha=1}^{M^2} (\alpha|\hat{A})|\alpha\rangle, & A_\alpha &:= (\alpha|\hat{A}). \end{aligned} \quad (2)$$

The relationship between the matrix elements A_{mn} and A_α depends on the choice of linear map transforming an $M \times M$ matrix \mathbf{A} into an $M^2 \times 1$ vector. Two such commonly used maps are given by [28]

$$\begin{aligned} \text{col } \mathbf{A} &:= \text{vec } \mathbf{A} = [A_{11}, \dots, A_{M1}, A_{12}, \dots, A_{M2}, \dots, A_{1M}, \dots, A_{MM}]^T, \\ \text{row } \mathbf{A} &:= \text{vec } \mathbf{A}^T = [A_{11}, \dots, A_{1M}, A_{21}, \dots, A_{2M}, \dots, A_{M1}, \dots, A_{MM}]^T, \end{aligned} \quad (3)$$

where the former stacks the columns of \mathbf{A} from left to right and the latter stacks the rows of \mathbf{A} from top to bottom. In order to illustrate (2) and (3) we consider a 2×2 matrix and its row representation:

$$\mathbf{A} = \begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix} \xrightarrow{\text{row}} \begin{bmatrix} (22|\hat{A}) \\ (21|\hat{A}) \\ (12|\hat{A}) \\ (11|\hat{A}) \end{bmatrix} = \begin{bmatrix} (1|\hat{A}) \\ (2|\hat{A}) \\ (3|\hat{A}) \\ (4|\hat{A}) \end{bmatrix}. \quad (4)$$

Linear maps $\check{T} : \mathcal{L}_{M^2} \rightarrow \mathcal{L}_{M^2}$ are usually called superoperators and represent the transformation $\check{T}|\hat{A}\rangle = |\check{B}\rangle$. Their matrix representation as an $M^2 \times M^2$ matrix $\mathbf{T} = [T_{\alpha\beta}]$ is defined as:

$$\check{T} = \sum_{\alpha, \beta=1}^{M^2} (\alpha|\check{T}|\beta)|\alpha\rangle\langle\beta|, \quad T_{\alpha\beta} := (\alpha|\check{T}|\beta), \quad (5)$$

where $|\alpha\rangle\langle\beta|$ is an orthonormal basis in the space of superoperators, induced by the basis in Liouville space. For consistency \mathbf{T} will be called a supermatrix. Every

superoperator may be written as [29]

$$\check{T} = \sum_{ij} t_{ij} \check{V}_i^L \check{V}_j^R, \quad (6)$$

where $\check{V}_i^L|\hat{A}\rangle = |\hat{V}_i\hat{A}\rangle$, $\check{V}_j^R|\hat{A}\rangle = |\hat{A}\hat{V}_j\rangle$ and $\{\hat{V}_r, r = 1, 2, \dots, M^2\}$ is a complete set of basis operators. Therefore, a general operator transformation is of the form

$$\check{T}|\hat{A}\rangle = \sum_{ij} t_{ij} |\hat{V}_i\hat{A}\hat{V}_j\rangle. \quad (7)$$

The linear maps in (3) induce two matrix representations of $\mathbf{V}_i\mathbf{A}\mathbf{V}_j$ [30]:

$$\begin{aligned} \text{col } \mathbf{V}_i\mathbf{A}\mathbf{V}_j &= (\mathbf{V}_j^T \otimes \mathbf{V}_i) \text{col } \mathbf{A}, & \mathbf{T}_{\text{col}} &= \mathbf{V}_j^T \otimes \mathbf{V}_i, \\ \text{row } \mathbf{V}_i\mathbf{A}\mathbf{V}_j &= (\mathbf{V}_i \otimes \mathbf{V}_j^T) \text{row } \mathbf{A}, & \mathbf{T}_{\text{row}} &= \mathbf{V}_i \otimes \mathbf{V}_j^T. \end{aligned} \quad (8)$$

We consider now the Hilbert space associated with a collection of N M -level systems $\mathcal{H}_M^{\otimes N} := \mathcal{H}_M^{(1)} \otimes \mathcal{H}_M^{(2)} \otimes \dots \otimes \mathcal{H}_M^{(N)}$. The basis of \mathcal{H}_M induces a basis of this space given by the ordered set $\{|h_1\rangle|h_2\rangle \dots |h_N\rangle, h_i = 1, 2, \dots, M\}$. The dimension of $\mathcal{H}_M^{\otimes N}$ is therefore M^N . The associated Liouville space is defined as $\mathcal{L}_{M^2}^{\otimes N} := \mathcal{L}_{M^2}^{(1)} \otimes \mathcal{L}_{M^2}^{(2)} \otimes \dots \otimes \mathcal{L}_{M^2}^{(N)}$. The basis of \mathcal{L}_{M^2} induces a basis of this space given by the ordered set $\{|l_1l'_1\rangle|l_2l'_2\rangle \dots |l_Nl'_N\rangle, l_i, l'_i = 1, 2, \dots, M\}$. The dimension of $\mathcal{L}_{M^2}^{\otimes N}$ is therefore M^{2N} .

3. Lie-algebraic solution of the Lindblad equation

In this section we describe a known algebraic method for finding an analytic solution to the initial value problem

$$\frac{d}{dt}\hat{\rho} = \check{L}\hat{\rho}, \quad \hat{\rho}(0) = \hat{\rho}_0, \quad (9)$$

where \check{L} is a time-independent linear combination of the generators of a finite-dimensional Lie algebra. Therefore, the formal solution to (9) is given by $\hat{\rho}(t) = \exp(\check{L}t)\hat{\rho}_0$.

The exponential e^{tA} of an element A of a finite-dimensional Lie algebra spanned by a set of n generators $\{L_j\}$, yields an element \mathcal{A} of the corresponding Lie group, which is parametrized in terms of *canonical coordinates of the first kind* [31]:

$$\mathcal{A}_1(\boldsymbol{\alpha}) = \exp \left[t \sum_{j=1}^n \alpha_j L_j \right]. \quad (10)$$

Another common parametrization involves *canonical coordinates of the second kind*:

$$\mathcal{A}_2(\boldsymbol{\beta}) = \prod_{j=1}^n \exp(\beta_j(t) L_j). \quad (11)$$

The parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}(t)$ are related through analytic expressions called Baker-Campbell-Hausdorff (BCH) formulas [32]. These formulas are usually obtained by solving a set of n coupled, nonlinear, ordinary differential equations, called Wei-Norman equations [33]. Recently, it was shown [34] that for the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ this nonlinear system can be reduced to a hierarchy of matrix Riccati equations. While it is known that matrix Riccati equations of the projective type with constant coefficients may be readily integrated [35], one must bear in mind that there is no

general method for obtaining the solution of a Riccati equation with time-dependent coefficients. However, approximate solutions can be obtained using the Taylor matrix method [36] and the variational iteration method [37]. Moreover, the number of Weierstrass equations that have to be integrated may be reduced [38].

Obtaining BCH formulas is simple in certain cases. It is known that they can always be obtained for solvable Lie algebras [39, 40]. Moreover, for small dimensions Riccati equations can be avoided, since calculation [41] of the matrix exponentials in (10) and (11) yields a system of algebraic nonlinear equations, that can be solved for the coefficients $\beta(t)$. In section 6 we will make use of this method. We remark that since the product of exponentials (11) is –in general– not global [39], care must be taken with the domain of validity of the BCH formulas obtained by any method.

4. The symmetric subspace of Liouville space

We call a state operator that is invariant under the exchange of the labels associated with individual systems a *symmetric state operator* and denote it $|\hat{\rho}_{\text{sym}}\rangle$. Since symmetric state operators are elements of $\mathcal{S}(\mathcal{L}_{M^2}^{\otimes N})$, the symmetric subspace of the Liouville space, the action of a symmetric Liouvillian \check{L}_{sym} on $|\hat{\rho}_{\text{sym}}\rangle$ will result in another element of $\mathcal{S}(\mathcal{L}_{M^2}^{\otimes N})$. That is, $\check{L}_{\text{sym}} : \mathcal{S}(\mathcal{L}_{M^2}^{\otimes N}) \rightarrow \mathcal{S}(\mathcal{L}_{M^2}^{\otimes N})$.

A basis vector of $\mathcal{S}(\mathcal{L}_{M^2}^{\otimes N})$ is given by

$$|S\rangle_{\{n_{ij}\}} = K \sum_P \check{P} \bigotimes_{i,j=1}^M |ij\rangle^{\otimes n_{ij}}, \quad K = \frac{1}{N!} \prod_{i,j=1}^M n_{ij}!, \quad (12)$$

where $|ij\rangle$ are unit basis vectors in \mathcal{L}_{M^2} , $n_{ij} = 0, 1, \dots, N$, $\sum_{i,j=1}^M n_{ij} = N$ and for nonzero n_{ij} , $|ij\rangle^{\otimes n_{ij}} = \bigotimes_{k=1}^{n_{ij}} |ij\rangle_k$. In (12) the index P runs over all permutations of two vectors $|ij\rangle, |i'j'\rangle$ that yield distinct tensor products and \check{P} is a superoperator that performs a given permutation. The constant K accounts for the permutations that yield repeated tensor products and ensures that the symmetric basis vectors have unit norm.

For three-level ($|0\rangle, |1\rangle, |2\rangle$) systems we introduce the following notation for symmetric basis vectors

$$|S\rangle_{\{n_{ij}\}} := Q_{\begin{smallmatrix} n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} & n_{22} \end{smallmatrix}} \quad (13)$$

that will be convenient in section 6. In (13), we have arbitrarily chosen n_{02} to depend on both the other indices and N and have thus omitted it. To illustrate (13) we consider $N = 3$, $n_{01} = 2$ and $n_{22} = 1$:

$$Q_{\begin{smallmatrix} 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}} = \frac{1}{3} [|22\rangle_1 |01\rangle_2 |01\rangle_3 + |01\rangle_1 |01\rangle_2 |22\rangle_3 + |01\rangle_1 |22\rangle_2 |01\rangle_3]. \quad (14)$$

We remark that some of the symmetric basis vectors are devoid of physical meaning, since they do not satisfy the unit trace condition.

The number of sets $\{n_{ij}\}$ that characterize all possible symmetric basis vectors is the dimension of $\mathcal{S}(\mathcal{L}_{M^2}^{\otimes N})$. It may be calculated as the number of ways to distribute N objects in M^2 bins, considering that any bin can be empty:

$$s := \binom{N + M^2 - 1}{N} = \frac{1}{(M^2 - 1)!} \prod_{k=0}^{M^2-2} (N + M^2 - 1 - k). \quad (15)$$

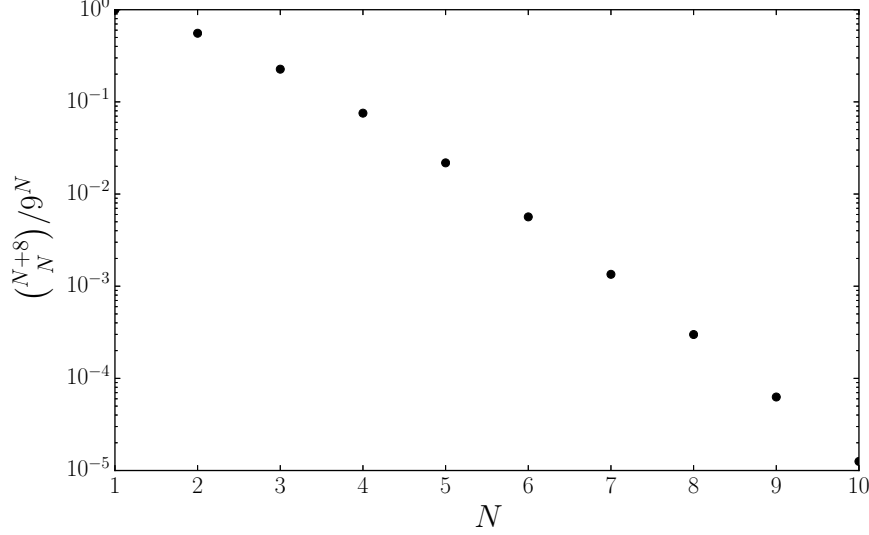


Figure 1. Comparison between the dimensions of the Liouville space and its symmetric subspace for N three-level systems.

Therefore, grouping the basis vectors of the Liouville space $\mathcal{L}_{M^2}^{\otimes N}$ into symmetric linear combinations results in a subspace whose dimension is polynomial in N .

In terms of symmetric basis vectors, a symmetric state operator is expressed as

$$|\hat{\rho}_{\text{sym}}) = \sum_{k=1}^s c_k |S_{n_k}), \quad (16)$$

where the c_k must be such that $\hat{\rho}_{\text{sym}}$ has the properties of a state operator and n_k denotes one of the s sets $\{n_{ij}\}$. From (15) it is clear that $|\hat{\rho}_{\text{sym}})$ has only $O(N^{M^2-1})$ independent parameters and therefore the computational resources required for the simulation of the symmetric Lindblad equation are substantially reduced compared to a simulation in the whole Liouville space (see Figure 1).

5. Symmetric Liouvillians with independent dissipative processes

In the following we shall be concerned with the algebraic structure of symmetric Liouvillians with independent dissipative processes (SInDiP Liouvillians) defined as:

$$\check{L}(\hat{\rho}) = \sum_{\mu=1}^N \check{L}^{\mu}(\hat{\rho}) = -\frac{i}{\hbar} \sum_{\mu=1}^N \check{H}^{\mu}(\hat{\rho}) + \sum_{\mu=1}^N \check{D}^{\mu}(\hat{\rho}), \quad (17)$$

with the superoperators

$$\check{H}^{\mu}(\hat{\rho}) = \left[\sum_{k=1}^{M^2-1} h_k \hat{F}_k^{\mu}, \hat{\rho} \right] = \sum_{k=1}^{M^2-1} h_k \left([\hat{F}_k^{\mu}]^L \mathbb{1}^R - \mathbb{1}^L [\hat{F}_k^{\mu}]^R \right) \hat{\rho} \quad (18)$$

and

$$\check{D}^{\mu}(\hat{\rho}) = \frac{1}{2} \sum_{j,k=1}^{M^2-1} a_{jk} (2\hat{F}_j^{\mu} \hat{\rho} \hat{F}_k^{\dagger\mu} - \hat{F}_k^{\dagger\mu} \hat{F}_j^{\mu} \hat{\rho} - \hat{\rho} \hat{F}_k^{\dagger\mu} \hat{F}_j^{\mu}) \quad (19)$$

$$= \frac{1}{2} \sum_{j,k=1}^{M^2-1} a_{jk} \left(2[\hat{F}_j^\mu]^L [\hat{F}_k^{\dagger\mu}]^R - [\hat{F}_k^{\dagger\mu} \hat{F}_j^\mu]^L \mathbf{1}^R - \mathbf{1}^L [\hat{F}_k^{\dagger\mu} \hat{F}_j^\mu]^R \right) \hat{\rho},$$

where we used one of the standard forms of the dissipator \check{D} , which is equivalent to the diagonal form used in (1). The coefficients h_k in (18) have to ensure the Hermiticity of the Hamiltonian and the coefficients a_{jk} in (19) are the entries of a complex, Hermitian, positive-semidefinite matrix. The operators $\hat{F}_j, j = 1, 2, \dots, M^2 - 1$ are traceless, orthonormal (with respect to the Hilbert-Schmidt inner product) and form a complete set. Since they are not necessarily Hermitian, they can be regarded as elements of the operator realization of the complex Lie algebra $\mathfrak{sl}(M)$ of traceless $M \times M$ matrices. The properties of the operators \hat{F}_j ensure that the decomposition of the generator \check{L} of a Markovian master equation into a Hamiltonian and a dissipator is unique [3].

Following [42] we use the term *strictly local operators* for the elements of the set

$$\zeta = \left\{ \mathbf{1}^1 \otimes \mathbf{1}^2 \otimes \dots \otimes \hat{O}_k^\mu \otimes \dots \otimes \mathbf{1}^N, \mu = 1, 2, \dots, N; k = 1, 2, \dots, M^2 - 1 \right\}, \quad (20)$$

where \hat{O}_k^μ acts on the μ -th system. The operator transformation $\check{L}(\hat{\rho})$ is a sum of N strictly local terms $\check{L}^\mu(\hat{\rho})$. Therefore we may write (17) in terms of the *collective superoperators* ($q = 1, 2, \dots, (n^2 - n)/2$ and $p = 1, 2, \dots, n - 1$)

$$\check{A}_\pm^q := \sum_{\mu=1}^N (\mathbb{A}_\pm^q)^\mu, \quad \check{A}_3^p := \sum_{\mu=1}^N (\mathbb{A}_3^p)^\mu, \quad (21)$$

where

$$(\mathbb{A}_\pm^q)^\mu = \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}}_{\mu-1} \otimes \mathbb{A}_\pm^q \otimes \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_{N-\mu}, \quad (22)$$

$$(\mathbb{A}_3^p)^\mu = \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}}_{\mu-1} \otimes \mathbb{A}_3^p \otimes \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_{N-\mu} \quad (23)$$

and the superoperators \mathbb{A}_\pm^q and \mathbb{A}_3^p are a realization of the generators of $\mathfrak{sl}(n = M^2)$.

In order to find an analytic expression for the symmetric state operator at time t , it will be necessary to calculate the action of the Liouvillian on the symmetric basis vectors (12). We recall from the theory of angular momentum that it is easier to calculate the matrix elements of operators acting on the symmetric representation of $\mathfrak{u}(n)$ if the operators are written in terms of the Jordan-Schwinger boson representation. The bosonization of a generator \mathbf{O}_α of a Lie algebra of dimension n is accomplished with the map [43]:

$$\mathcal{B}(\mathbf{O}_\alpha) := \sum_{j,k=1}^n \hat{b}_j^\dagger (\mathbf{O}_\alpha)_{jk} \hat{b}_k, \quad (24)$$

where the operators \hat{b} satisfy the usual bosonic commutation relations $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}$ and $[\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0$, and the products $\hat{b}_i^\dagger \hat{b}_j$ satisfy the commutation relations of the generators of the algebra $\mathfrak{u}(n)$:

$$[\hat{b}_i^\dagger \hat{b}_j, \hat{b}_m^\dagger \hat{b}_n] = \hat{b}_i^\dagger \hat{b}_n \delta_{jm} - \hat{b}_m^\dagger \hat{b}_j \delta_{in}. \quad (25)$$

The map (24) has the property of preserving all commutators, $\mathcal{B}([\mathbf{O}_\alpha, \mathbf{O}_\beta]) = [\mathcal{B}(\mathbf{O}_\alpha), \mathcal{B}(\mathbf{O}_\beta)]$. Therefore, it is a Lie algebra homomorphism. We remark that this procedure is only introduced as an algebraic device. The boson operators should not

be understood in the sense of quantum field theory. We also note that in the case of Liouville space, the boson operators are actually superoperators acting on the unit basis vectors $|ij\rangle$ defined in section 4.

In order to better understand the bosonization procedure in the present context, we consider the superoperator $\hat{\sigma}_{20,+}^L \mathbb{1}^R$ acting on a three-level system. The nonzero matrix elements of this superoperator read:

$$(20|\hat{\sigma}_{20,+}^L \mathbb{1}^R|00) = (21|\hat{\sigma}_{20,+}^L \mathbb{1}^R|01) = (22|\hat{\sigma}_{20,+}^L \mathbb{1}^R|02) = 1, \quad (26)$$

where we used (7) to calculate $\hat{U}^L \hat{V}^R |ij\rangle = |\hat{U}|i\rangle \langle j|\hat{V}\rangle = |i'j'\rangle$. Therefore, the bosonization mapping yields

$$\hat{\sigma}_{20,+}^L \mathbb{1}^R \mapsto \check{b}_{20}^\dagger \check{b}_{00} + \check{b}_{21}^\dagger \check{b}_{01} + \check{b}_{22}^\dagger \check{b}_{02}. \quad (27)$$

In terms of bosonic superoperators, the collective superoperators defined in (21) have the form ($\ell_i > \ell_j$)

$$\check{A}_+^{\ell_i \ell_j} = \sum_{\mu=1}^N \check{b}_{\ell_i}^{\dagger(\mu)} \check{b}_{\ell_j}^{(\mu)}, \quad \check{A}_-^{\ell_i \ell_j} = \sum_{\mu=1}^N \check{b}_{\ell_j}^{\dagger(\mu)} \check{b}_{\ell_i}^{(\mu)}, \quad \check{A}_0^{\ell_i \ell_j} = \sum_{\mu=1}^N \frac{1}{2} \left(\check{b}_{\ell_i}^{\dagger(\mu)} \check{b}_{\ell_i}^{(\mu)} - \check{b}_{\ell_j}^{\dagger(\mu)} \check{b}_{\ell_j}^{(\mu)} \right), \quad (28)$$

where ℓ_i and ℓ_j denote the labels of two unit basis vectors $|ii'\rangle$ and $|jj'\rangle$. We remark that the elements of the set $\{\check{A}_0^{\ell_i \ell_j}\}$ are not linearly independent and, therefore, it is necessary to choose a subset thereof, denoted as $\{\check{A}_3^{\ell_i \ell_j}\}$, comprising $M^2 - 1$ superoperators. Moreover, using (25) it follows that the collective superoperators (28) form $\mathfrak{sl}(2)$ subalgebras of $\mathfrak{sl}(M^2)$:

$$\check{A}_0^{\ell_i \ell_j} := \frac{1}{2} [\check{A}_+^{\ell_i \ell_j}, \check{A}_-^{\ell_i \ell_j}], \quad [\check{A}_0^{\ell_i \ell_j}, \check{A}_\pm^{\ell_i \ell_j}] = \pm \check{A}_\pm^{\ell_i \ell_j}. \quad (29)$$

This realization of the generators of $\mathfrak{sl}(M^2)$ simplifies the calculation of the action of a collective superoperator on a symmetric vector. For example, using the notation introduced in (13) for three-level systems:

$$\begin{aligned} \check{A}_+^{22} Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} & n_{22}}} &= n_{21} Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} - 1 & n_{22} + 1}}, \\ \check{A}_-^{22} Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} & n_{22}}} &= n_{22} Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} + 1 & n_{22} - 1}}, \\ \check{A}_0^{22} Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} & n_{22}}} &= \frac{1}{2} (n_{22} - n_{21}) Q_{\substack{n_{00} & n_{01} \\ n_{10} & n_{11} & n_{12} \\ n_{20} & n_{21} & n_{22}}}. \end{aligned} \quad (30)$$

In the following section we will make use of the main results of this article, contained in this section and the previous section, in order to obtain an analytic expression for the solution of a Lindblad equation with a SInDiP Liouvillian. For quick reference we outline the procedure here:

- Starting from a master equation in the usual notation (17) bosonize each superoperator of the form $\hat{U}^L \hat{V}^R$ and write the master equation in terms of the collective superoperators in (28).
- Write the solution of the master equation as a product of, in general entangled, exponentials remembering that $e^{A+B} = e^A e^B$ iff $[A, B] = 0$. Obtain the appropriate BCH formulas that allow disentangling the exponentials.
- Calculate the action of the superoperator $e^{\check{L}t}$ on a symmetric basis vector.

6. Example: three-level atoms

In this section we will use the results of the preceding sections to calculate an analytic expression for the symmetric state operator of a collection of N three-level atoms in the Λ configuration with energy levels $E_0 < E_1 < E_2$, interacting with a radiation bath. Assuming orthogonal dipole moments and neglecting the environment-mediated coupling between atoms, the corresponding Lindblad master equation is given by [44]

$$\frac{d}{dt}\hat{\rho} = \check{L}(\hat{\rho}) = \sum_{\mu=1}^N \check{L}^{\mu}(\hat{\rho}) = \sum_{\mu=1}^N -\frac{i}{\hbar}[\hat{H}^{\mu}, \hat{\rho}] + (\check{L}_{21}^{\mu} + \check{L}_{20}^{\mu})\hat{\rho}, \quad (31)$$

with the Lindbladians

$$\begin{aligned} \check{L}_{ij}^{\mu}[\bullet] = & (N_0 + 1)\gamma_{ij} \left(\hat{\sigma}_{ij-}^{\mu}[\bullet]\hat{\sigma}_{ij+}^{\mu} - \frac{1}{2}\{\hat{\sigma}_{ij+}^{\mu}\hat{\sigma}_{ij-}^{\mu}, [\bullet]\} \right) \\ & + N_0\gamma_{ij} \left(\hat{\sigma}_{ij+}^{\mu}[\bullet]\hat{\sigma}_{ij-}^{\mu} - \frac{1}{2}\{\hat{\sigma}_{ij-}^{\mu}\hat{\sigma}_{ij+}^{\mu}, [\bullet]\} \right), \end{aligned} \quad (32)$$

where $\{A, B\} = AB + BA$ and $[\bullet]$ is a placeholder for the operand of the superoperator. The Hamiltonian \hat{H}^{μ} , taking $\bar{E} = \frac{1}{3}(E_0 + E_1 + E_2)$ as the zero of energy, is

$$\hat{H}^{\mu} = \tilde{E}_0\hat{\sigma}_3^{02(\mu)} + \tilde{E}_1\hat{\sigma}_3^{12(\mu)}, \quad \tilde{E}_i = 2(\bar{E} - E_i). \quad (33)$$

The index μ denotes that an operator acts on the Hilbert space of the atom with label μ and we define $\hat{\sigma}_{ij-} = |j\rangle\langle i|$ and $\hat{\sigma}_3^{ij} = \frac{1}{2}(|j\rangle\langle j| - |i\rangle\langle i|)$. Moreover, the products $\hat{\sigma}_{ij-}\hat{\sigma}_{ij+} = |j\rangle\langle j|$ may be written as [45]:

$$|j\rangle\langle j| = \frac{1}{3} + 2 \sum_{k=0}^{j-1} \hat{\sigma}_3^{k,k+1} - \frac{2}{3} \sum_{k=0}^1 (2-k)\hat{\sigma}_3^{k,k+1}. \quad (34)$$

Since the operators $\{\hat{\sigma}_{\pm}^{21}, \hat{\sigma}_{\pm}^{20}, \hat{\sigma}_{\pm}^{10}, \hat{\sigma}_3^{12}, \hat{\sigma}_3^{02}\}$ are a realization of the generators of $\mathfrak{sl}(3)$, in general, we can construct 108 bosonized collective superoperators $\check{A}_{\pm,0}^{\ell_i\ell_j}$ with $\ell_i, \ell_j = 00, 01, 02, 10, 11, 12, 20, 21, 22$. However, as pointed out in section 5, from the set of 36 superoperators $\check{A}_0^{\ell_j}$ one must choose a subset of eight linearly independent superoperators $\{\check{A}_3^{\ell_i}\}$. The set $\{\check{A}_{\pm}^{\ell_i}\} \cup \{\check{A}_3^{\ell_j}\}$ is a realization of the generators of $\mathfrak{sl}(9)$ and its elements are shown in the appendix.

In terms of bosonized collective superoperators, (31) is written as

$$\begin{aligned} \frac{d}{dt}|\hat{\rho}\rangle = & \left[N_0\gamma_{20}\check{S}_+^{20} + N_0\gamma_{21}\check{S}_+^{21} + \gamma_{20}N_0^+\check{S}_-^{20} + \gamma_{21}N_0^+\check{S}_-^{21} + \frac{2}{3}\left(N_0^-\gamma_{20} - \gamma_{21}\tilde{N}_0\right)\check{S}_3^{20} \right. \\ & + \frac{2}{3}\left(N_0^-\gamma_{21} - \gamma_{20}\tilde{N}_0\right)\check{S}_3^{21} + 3\frac{i}{\hbar}\tilde{E}_0(\check{A}_3^{10} - \check{A}_3^{01}) + 2\frac{i}{\hbar}(E_2 - E_0)(\check{A}_3^{02} - \check{A}_3^{20}) \\ & + \frac{1}{3}\left(N_0^-\gamma_{20} - \gamma_{21}\tilde{N}_0\right)(\check{A}_3^{10} + \check{A}_3^{01}) + \frac{1}{3}\left(N_0^-\gamma_{21} - \gamma_{20}\tilde{N}_0\right)(\check{A}_3^{02} + \check{A}_3^{20}) \\ & \left. - 4\frac{i}{\hbar}(2E_0 + E_2 - 3\bar{E})\check{A}_3^{12} - \frac{1}{3}\gamma\tilde{N}_0N\mathbf{1} \right]|\hat{\rho}\rangle := \check{L}|\hat{\rho}\rangle, \end{aligned} \quad (35)$$

where $\gamma = \gamma_{20} + \gamma_{21}$, $\tilde{N}_0 = 2N_0 + 1$, $N_0^{\pm} = N_0 \pm 1$, and we defined

$$\check{S}_{\pm,3}^{21} := \check{A}_{\pm,3}^{22}, \quad \check{S}_{\pm,3}^{20} := \check{A}_{\pm,3}^{22}, \quad \check{S}_{\pm,3}^{10} := \check{A}_{\pm,3}^{11}. \quad (36)$$

The set consisting of eight linearly independent superoperators

$$\{\check{A}_3^{\ell_j}\} := \{\check{A}_3^{22}, \check{A}_3^{12}, \check{A}_3^{10}, \check{A}_3^{02}\} \cup \{\check{A}_3^{22}, \check{A}_3^{21}, \check{A}_3^{12}, \check{A}_3^{11}\} \quad (37)$$

was constructed considering two quartets (sets consisting of four elements) of $\mathfrak{sl}(2)$ subalgebras of $\mathfrak{sl}(9)$, such that in each quartet the generators of any algebra commute with the generators of the other algebras. We remark that any such quartet may be used to label the symmetric basis vectors in terms of the eigenvalues of the Cartan and Casimir superoperators of the algebras.

Since \check{L} is time-independent, the formal solution of (35) is given by

$$|\hat{\rho}(t)\rangle = e^{\alpha_1 t} \exp(\alpha_2 \check{A}_3^{21} t) \exp(\alpha_3 \check{A}_3^{01} t) \exp(\alpha_4 \check{A}_3^{20} t) \exp(\alpha_5 \check{A}_3^{12} t) \exp(\alpha_6 \check{A}_3^{02} t) \cdot \exp[(\alpha_3^{21} \check{S}_3^{21} + \alpha_3^{20} \check{S}_3^{20} + \alpha_-^{21} \check{S}_-^{21} + \alpha_-^{20} \check{S}_-^{20} + \alpha_+^{21} \check{S}_+^{21} + \alpha_+^{20} \check{S}_+^{20})t] |\hat{\rho}(0)\rangle, \quad (38)$$

where $\alpha_1, \dots, \alpha_6$ are the coefficients of the corresponding superoperators in (35). Since the superoperators in (36) –excluding \check{S}_3^{10} – are a realization of the generators of $\mathfrak{sl}(3)$, we may disentangle the last exponential in (38) as

$$\exp\left[\left(\sum_{\ell_i} \alpha_+^{\ell_i} \check{S}_+^{\ell_i} + \sum_{\ell_j} \alpha_3^{\ell_j} \check{S}_3^{\ell_j} + \sum_{\ell_k} \alpha_-^{\ell_k} \check{S}_-^{\ell_k}\right)t\right] = \prod_{\ell_i} \exp(\beta_+^{\ell_i}(t) \check{S}_+^{\ell_i}) \prod_{\ell_j} \exp(\beta_3^{\ell_j}(t) \check{S}_3^{\ell_j}) \prod_{\ell_k} \exp(\beta_-^{\ell_k}(t) \check{S}_-^{\ell_k}), \quad (39)$$

where $\ell_i \in (20, 10, 21)$, $\ell_j \in (21, 20)$ and $\ell_k \in (21, 10, 20)$. The corresponding BCH formulas are obtained calculating the exponentials in (39) (using the 3×3 matrix representation of $\mathfrak{sl}(3)$) and solving a system of nonlinear equations:

$$\begin{aligned} e^{-\beta_3^{21}(t)} &= -e^{-\beta_3^{20}(t)} \beta_+^{10}(t) \beta_-^{10}(t) + \frac{1}{D} [-\alpha_3^{21} f_1(t) + f_0(t) + f_2(t) (\alpha_+^{21} \alpha_-^{21} + (\alpha_3^{21})^2)], \\ e^{-\beta_3^{20}(t)} &= \frac{1}{D} [-\alpha_3^{20} f_1(t) + f_0(t) + f_2(t) (\alpha_+^{20} \alpha_-^{20} + (\alpha_3^{20})^2)], \\ \beta_+^{21}(t) &= e^{\beta_3^{21}(t)} \left\{ -\beta_-^{10}(t) \beta_+^{20}(t) e^{-\beta_3^{20}(t)} + \frac{\alpha_+^{21}}{D} [\alpha_3^{20} f_2(t) + f_1(t)] \right\}, \\ \beta_-^{21}(t) &= e^{\beta_3^{21}(t)} \left\{ -\beta_+^{10}(t) \beta_-^{20}(t) e^{-\beta_3^{20}(t)} + \frac{\alpha_-^{21}}{D} [\alpha_3^{20} f_2(t) + f_1(t)] \right\}, \\ \beta_+^{10}(t) &= \frac{\alpha_+^{20} f_2(t)}{D} \alpha_-^{21} e^{\beta_3^{20}(t)}, & \beta_+^{20}(t) &= \frac{\alpha_+^{20} e^{\beta_3^{20}(t)}}{D} [\alpha_3^{21} f_2(t) + f_1(t)], \\ \beta_-^{10}(t) &= \frac{\alpha_-^{20} f_2(t)}{D} \alpha_+^{21} e^{\beta_3^{20}(t)}, & \beta_-^{20}(t) &= \frac{\alpha_-^{20} e^{\beta_3^{20}(t)}}{D} [\alpha_3^{21} f_2(t) + f_1(t)], \end{aligned} \quad (40)$$

with (λ, μ, ν) are the eigenvalues of the matrix in the argument of the exponential in the l.h.s of (39)

$$\begin{aligned} D &= (\mu - \lambda)(\nu - \lambda)(\mu - \nu), \\ f_0(t) &= (\mu^2 \nu - \mu \nu^2) e^{\lambda t} + (\nu^2 \lambda - \nu \lambda^2) e^{\mu t} + (\lambda^2 \mu - \lambda \mu^2) e^{\nu t}, \\ f_1(t) &= (\nu^2 - \mu^2) e^{\lambda t} + (\lambda^2 - \nu^2) e^{\mu t} + (\mu^2 - \lambda^2) e^{\nu t}, \\ f_2(t) &= (\mu - \nu) e^{\lambda t} + (\nu - \lambda) e^{\mu t} + (\lambda - \mu) e^{\nu t}. \end{aligned} \quad (41)$$

If the radiation bath is in the ground state ($N_0 = 0$, $\mu = \nu$),

$$D = \lambda - \mu, \quad f_0(t) = \lambda e^{\mu t} - \mu e^{\lambda t}, \quad f_1(t) = e^{\lambda t} - e^{\mu t}, \quad f_2(t) = 0,$$

and the BCH relations simplify considerably:

$$\begin{aligned} \beta_3^{20}(t) &= \beta_3^{21}(t) = -2\gamma t/3, & \beta_+^{21} &= \beta_+^{20} = \beta_{\pm}^{10} = 0, \\ \beta_-^{21}(t) &= \frac{\gamma^{21}}{\gamma} (1 - e^{-\gamma t}), & \beta_-^{20}(t) &= \frac{\gamma^{20}}{\gamma} (1 - e^{-\gamma t}). \end{aligned} \quad (42)$$

Since an arbitrary, symmetric initial state operator is a linear combination of symmetric vectors (16), in order to obtain an expression for $|\hat{\rho}(t)\rangle$ we need to calculate:

$$\begin{aligned} e^{\check{L}t}|S_{n_k}\rangle &= \sum_{i_+=0}^{n_{00}} \binom{n_{00}+i_-+j_- - j_+}{i_+} (\beta_+^{20})^{i_+} \sum_{j_+=0}^{n_{00}} \binom{n_{00}+i_-+j_-}{j_+} (\beta_+^{10})^{j_+} \\ &\times \sum_{k_+=0}^{n_{11}} \binom{n_{11}-j_-+k_-}{k_+} (\beta_+^{21})^{k_+} \exp(\beta_3^{21} \tilde{a}_3^{21}) \exp(\beta_3^{20} \tilde{a}_3^{20}) \sum_{k_-=0}^{n_{22}} \binom{n_{22}-i_-}{k_-} (\beta_-^{21})^{k_-} \\ &\times \sum_{j_-=0}^{n_{11}} \binom{n_{11}}{j_-} (\beta_-^{10})^{j_-} \sum_{i_-=0}^{n_{22}} \binom{n_{22}}{i_-} (\beta_-^{20})^{i_-} C(t) \mathbf{Q}^{\nu_{00} \nu_{01} \nu_{10} \nu_{11} \nu_{12} \nu_{20} \nu_{21} \nu_{22}}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} C(t) &= e^{\alpha_1 t} \exp[\tfrac{1}{2}\alpha_6(N - \nu_{00} - \nu_{11} - \nu_{22} - n_{10} - n_{20} - n_{21} - n_{12} - 2n_{01})t] \times \\ &\exp[\tfrac{1}{2}\alpha_2(n_{21} - n_{12})t] \exp[\tfrac{1}{2}\alpha_3(n_{21} - n_{01})t] \exp[\tfrac{1}{2}\alpha_4(n_{20} - n_{10})t] \exp[\tfrac{1}{2}\alpha_5(n_{12} - n_{10})t], \\ \tilde{a}_3^{21} &= [n_{22} - (n_{11} - j_- + k_-)]/2, \\ \tilde{a}_3^{20} &= [n_{22} - (n_{00} + i_- + j_-)]/2, \\ \nu_{00} &= n_{00} + i_- + j_- - j_+ - i_+ \\ \nu_{11} &= n_{11} - j_- + k_- - k_+ + j_+ \\ \nu_{22} &= n_{22} - i_- - k_- + k_+ + i_+. \end{aligned} \quad (44)$$

In summary, the analytic solution to the Lindblad master equation (31) is given by the expression:

$$\begin{aligned} |\hat{\rho}(t)\rangle &= e^{\alpha_1 t} \exp(\alpha_2 \check{A}_3^{21} t) \exp(\alpha_3 \check{A}_3^{01} t) \exp(\alpha_4 \check{A}_3^{20} t) \exp(\alpha_5 \check{A}_3^{12} t) \exp(\alpha_6 \check{A}_3^{02} t) \\ &\times \exp(\beta_3^{21}(t) \check{S}_3^{21}) \exp(\beta_3^{20}(t) \check{S}_3^{20}) \exp(\beta_-^{21}(t) \check{S}_-^{21}) \exp(\beta_-^{20}(t) \check{S}_-^{20}) \exp(\beta_+(t) \check{S}_+^{21}) \\ &\times \exp(\beta_+^{20}(t) \check{S}_+^{20}) |\hat{\rho}(0)\rangle, \end{aligned} \quad (45)$$

where the betas were calculated in (40) and the explicit expression for $|\hat{\rho}(t)\rangle$ can be obtained evaluating (43) for each of the component vectors of the initial state operator. To the best of our knowledge, this solution has not been obtained before.

It is worth pointing out that even though the superoperators describing the dynamics of a three-level system are generators of $\mathfrak{sl}(9)$, in many situations it is only necessary to disentangle an exponential of elements of a lower-dimensional algebra in order to obtain an analytic expression for $\hat{\rho}(t)$. In the example above we disentangled an exponential of elements of $\mathfrak{sl}(3)$.

In addition to the results presented above, disentangling (38) allows studying the dynamics of the open system in a basis of coherent states in Liouville space. This has been done recently for one two-level system interacting with a bosonic bath [46]. Working in the Heisenberg picture, the same procedure allows finding conserved quantities of the open system [47].

As a final remark, we would like to point out that in the case that the system parameters in the Lindblad equation are time-dependent, the tools developed here, namely the bosonized superoperators and the symmetric basis vectors, together with the Lie-algebraic method discussed in [48] enable finding a semi-analytic solution, that would involve numerically integrating a system of differential equations. Additionally, it would be feasible to study how the system approaches its equilibrium state.

7. Conclusions

In this work we considered the permutation-symmetric Lindblad equation describing the open-system dynamics of a collection of N M -level systems subject to independent dissipation processes. We constructed a basis of the symmetric subspace of Liouville space and showed that its dimension grows polynomially with the number of systems. Therefore, given a symmetric initial state operator, this result can be used to efficiently simulate any symmetric Lindblad equation, since the computational resources required are substantially reduced compared to a simulation in the whole Liouville space. We also built a set of superoperators, whose action on this basis is easily specified, that are generators of the Lie algebra $\mathfrak{sl}(M^2)$ and thus enable writing any such Lindblad equation. We showed that these results can be used to obtain an analytic solution of the Lindblad equation by means of Lie-algebraic methods.

In order to show the usefulness of these results, we calculated the evolution of the state operator of a collection of three-level atoms interacting with independent radiation baths. Moreover, even though the master equation is written in terms of elements of $\mathfrak{sl}(9)$, finding the solution required working only with an $\mathfrak{sl}(3)$ subalgebra.

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Appendix A. Bosonized collective superoperators, generators of $\mathfrak{sl}(9)$

The following table enables a researcher to quickly rewrite a master equation in a form suitable for application of the solution method outlined in section 5.

$\sum_i \hat{I}_3^{(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} = (\check{A}_{\mp}^{11} + \check{A}_{\mp}^{01} + \check{A}_{\mp}^{21}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\mp^{21(i)} = \check{A}_{\pm}^{22} \hat{\rho}$
$\sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{I}_3^{(i)} = (\check{A}_{\pm}^{12} + \check{A}_{\pm}^{01} + \check{A}_{\pm}^{10}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\mp^{20(i)} = \check{A}_{\pm}^{22} \hat{\rho}$
$\sum_i \hat{I}_3^{(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} = (\check{A}_{\mp}^{22} + \check{A}_{\mp}^{12} + \check{A}_{\mp}^{02}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\mp^{21(i)} = \check{A}_{\pm}^{22} \hat{\rho}$
$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{I}_3^{(i)} = (\check{A}_{\pm}^{21} + \check{A}_{\pm}^{10} + \check{A}_{\pm}^{22}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\mp^{20(i)} = \check{A}_{\pm}^{22} \hat{\rho}$
$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{I}_3^{(i)} = (\check{A}_{\pm}^{21} + \check{A}_{\pm}^{20} + \check{A}_{\pm}^{22}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} = \check{A}_{\pm}^{21} \hat{\rho}$
$\sum_i \hat{I}_3^{(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} = (\check{A}_{\mp}^{22} + \check{A}_{\mp}^{12} + \check{A}_{\mp}^{02}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\mp^{10(i)} = \check{A}_{\pm}^{21} \hat{\rho}$
$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} = \frac{1}{2} (\check{A}_{\pm}^{22} - \check{A}_{\pm}^{20}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} = \check{A}_{\pm}^{21} \hat{\rho}$
$\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} = \frac{1}{2} (\check{A}_{\mp}^{22} - \check{A}_{\mp}^{02}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\mp^{10(i)} = \check{A}_{\pm}^{21} \hat{\rho}$
$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} = \frac{1}{2} (\check{A}_{\pm}^{22} - \check{A}_{\pm}^{20}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} = \check{A}_{\pm}^{20} \hat{\rho}$
$\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} = \frac{1}{2} (\check{A}_{\mp}^{22} - \check{A}_{\mp}^{02}) \hat{\rho}$	$\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} = \check{A}_{\pm}^{20} \hat{\rho}$

$$\begin{aligned}
\sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} &= \frac{1}{2} (\check{A}_\pm^{12} - \check{A}_\pm^{10}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} &= \check{A}_\pm^{20} \hat{\rho} \\
\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} &= \frac{1}{2} (\check{A}_\mp^{21} - \check{A}_\mp^{00}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} &= \check{A}_\pm^{20} \hat{\rho} \\
\sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= \frac{1}{2} (\check{A}_\pm^{12} - \check{A}_\pm^{01}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\mp^{21(i)} &= \check{A}_\pm^{12} \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} &= \frac{1}{2} (\check{A}_\mp^{21} - \check{A}_\mp^{10}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\mp^{20(i)} &= \check{A}_\pm^{12} \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} &= \frac{1}{2} (\check{A}_\mp^{21} - \check{A}_\mp^{10}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} &= \check{A}_\pm^{12} \hat{\rho} \\
\sum_i \hat{\sigma}_\pm^{20(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= \frac{1}{2} (\check{A}_\pm^{22} - \check{A}_\pm^{01}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\mp^{10(i)} &= \check{A}_\pm^{11} \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{\sigma}_\pm^{21(i)} &= \frac{1}{2} (\check{A}_\mp^{21} - \check{A}_\mp^{11}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\pm^{20(i)} &= \check{A}_\pm^{10} \hat{\rho} \\
\sum_i \hat{\sigma}_\pm^{21(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= \frac{1}{2} (\check{A}_\pm^{22} - \check{A}_\pm^{11}) \hat{\rho} & \sum_i \hat{\sigma}_\pm^{10(i)} \hat{\rho} \hat{\sigma}_\pm^{10(i)} &= \check{A}_\pm^{10} \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} &= \frac{1}{2} (-\check{A}_3^{21} - \check{A}_3^{20} + \check{A}_3^{11} + \check{A}_3^{02} + \check{A}_3^{22} + \check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{I}_3^{(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} &= (-\check{A}_3^{21} - \check{A}_3^{20} + 2\check{A}_3^{12} + \check{A}_3^{02} + \check{A}_3^{22} + \check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= \frac{1}{2} (-\check{A}_3^{01} + \check{A}_3^{11} + \check{A}_3^{22}) \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{I}_3^{(i)} &= (\check{A}_3^{20} + \check{A}_3^{22} + \check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{I}_3^{(i)} &= (2\check{A}_3^{21} + \check{A}_3^{20} - \check{A}_3^{12} - \check{A}_3^{02} + \check{A}_3^{22} - \check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{I}_3^{(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= (\check{A}_3^{02} + \check{A}_3^{22} - \check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{\sigma}_3^{12(i)} \hat{\rho} \hat{\sigma}_3^{12(i)} &= (-\check{A}_3^{21} + \check{A}_3^{11} + \check{A}_3^{02} + \frac{1}{2}\check{A}_3^{22} + \frac{1}{2}\check{A}_3^{21}) \hat{\rho} \\
\sum_i \hat{\sigma}_3^{02(i)} \hat{\rho} \hat{\sigma}_3^{02(i)} &= \frac{1}{2} (\check{A}_3^{12} + \check{A}_3^{02} + \check{A}_3^{21} - \check{A}_3^{21} - \check{A}_3^{20} - \check{A}_3^{22}) \hat{\rho} + (\check{A}_3^{11} + \check{A}_3^{22}) \hat{\rho}
\end{aligned}$$

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